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Interlacing of Alternation Points in Rational Approximation

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1. INTRODUCTION

Let $R_m{}^n[a, b]$ denote the set of rational functions $r = p/q$ where p is a polynomial of degree $\leqslant n$, q is a polynomial of degree $\leqslant m$ and $q > 0$ on $[a, b]$. Let $k = n + m + 1$. For each continuous function f defined on $[a, b]$ let r_f denote the rational function from $R_m{}^n[a, b]$ which is the best approximation to f in the uniform (Chebyshev) norm, let $e_k = f - r_f$ and set $E_k(f) = \| e_k \|$. e_k is called the error curve of f and $E_k(f)$ the deviation. It is well known that if f is normal there exist at least $k + 1$ points $x_0 < \cdots < x_k$ in $[a, b]$ such that $e_k(x_i) = -e_k(x_{i+1}) = \pm E_k(f)$. These points will be called the alternation points of e_k. It follows that there exist at least k points $u_0 < \cdots < u_{k-1}$ which are zeros of e_k. These points will be called the interpolation points of f.

From the theory of rational interpolation it is known that if

$$r = p/q \in R_m{}^n[a, b]$$

interpolates $f \in C[a, b]$ at points $y_1 < \cdots < y_k$, where $k = n + m + 1$, and if $f^{(k)}$ exists on (a, b), then for each x in $[a, b]$,

$$f(x) - p(x)/q(x) = [((x - y_1) \cdots (x - y_k))/k!q(x)](fq)^{(k)}(\xi(x))$$

for some $\xi(x)$ in the interval determined by $y_1, \ldots, y_k$ and x. Therefore, if $r_f = p/q$ is the best approximation to f and f is normal and $u_0, \ldots, u_{k-1}$ are interpolation points of f, then

$$f(x) - r_f(x) = [((x - u_0) \cdots (x - u_{k-1}))/k!q(x)](fq)^{(k)}(\xi(x)),$$

where $x_j < u_j < x_{j+1}$, $j = 0, \ldots, k - 1$. If we assume that $(fq)^{(k)}$ does not change sign on (a, b), then it follows easily that e_k has precisely $k + 1$ alternation points by considering the error term for rational interpolation; and if $(fq)^{(k)} \geqslant 0$ on $[a, b]$, then $e_k(x_k) = +E_k(f)$.

Suppose $r_g = \bar{p}/\bar{q}$ is the best approximation to $g \in C[a, b]$ from $R_m^{n-1}[a, b](R_{m-1}^n[a, b])$. If g is normal and $(g\bar{q})^{(k-1)} \geqslant 0$ on (a, b), then e_{k-1} has exactly k alternation points $z_0 < \cdots < z_{k-1}$. It will be shown that under appropriate conditions the alternation points of e_{k-1} interlace those of e_k. This property has been established by Shohat [3] in the case of polynomial approximation. Recently, Rowland [2] proved that the interpolation points in polynomial approximation have the same property. It is the purpose of this paper to extend Shohat's result to the case of rational approximation

2. Main Result

Following the notation in Rowland [2], denote the j-th alternation points of e_k and e_{k-1} by x_j and z_j, respectively, starting with the subscript zero. Let A_+ denote the class of functions which are nonnegative on (a, b) and do not vanish on any subinterval of (a, b). Let $A_- = \{f\colon -f \in A_+\}$ and $A = A_+ \cup A_-$.

Theorem. *Suppose $r_f = p/q$, $r_g = \bar{p}/\bar{q}$ are the best approximations to f, $g \in C[a, b]$ from $R_m{}^n[a, b]$ and $R_m^{n-1}[a, b](R_{m-1}^n[a, b])$, respectively. Suppose further that f and g are normal and that $(fq)^{(k)} \in A_+$ and $(g\bar{q})^{(k-1)} \in A_+$, $k = n + m + 1$.*

$$\text{If } f^{(k)}/E_k(f) + g^{(k)}/E_{k-1}(g) \in A, \text{ then } x_0 < z_0 \text{ unless } a = x_0 = z_0; \quad x_j < z_j, \quad j = 1,\ldots, k-1 \tag{2.1}$$

$$\text{If } f^{(k)}/E_k(f) - g^{(k)}/E_{k-1}(g) \in A, \text{ then } z_j < x_{j+1}, \quad j = 1,\ldots, k-1; \quad z_k < x_{k+1} \text{ unless } z_k = x_{k+1} = b. \tag{2.2}$$

Proof. The proof will proceed as in [2]. Define functions F and G by

$$F = (f - r_f)/E_k(f), \qquad G = (g - r_g)/E_{k-1}(g).$$

Using the remainder formula for rational interpolation we see that $F(x_k) = G(z_{k-1}) = 1$. Therefore

$$F(x_j) = (-1)^{k-j}, \qquad j = 0,\ldots, k$$
$$G(z_j) = (-1)^{k-1-j}, \qquad j = 0,\ldots, k-1.$$

Define a function h by $h = F + G$. Since $\| F \| = \| G \| = 1$,

$$(-1)^{k-j} h(x_j) \geqslant 0, \qquad j = 0,\ldots, k \tag{2.3}$$
$$(-1)^{k-1-j} h(z_j) \geqslant 0, \qquad j = 0,\ldots, k-1. \tag{2.4}$$

We will need two results from [2].

(2.5) Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If $a_1 < \cdots < a_m$ are points in $[a, b]$ and $(-1)^j f(a_j) \geqslant 0$, $j = 1,..., m$, then f has at least $m - 1$ zeros on $[a, b]$.

(2.6) If f is continuous on $[a, b]$ and $f^{(k)} \in A$, then f has at most k zeros on $[a, b]$, counting interior double zeros twice.

To establish (2.1), assume the contrary and let i denote the first index for which $z_i \leqslant x_i$. We will show that

(2.7) h has at least i zeros on $[a, z_{i-1}]$, and

(2.8) h has at least $k + 1 - i$ zeros on $[z_i, b]$.

These two conditions will then contradict (2.6).

If $i = 0$ and $a < x_0 = z_0$, then h has a double zero at x_0 and (2.3) and (2.5) show that h has at least $k - 1$ zeros on $[x_1, x_k]$. Thus h has at least $k + 1$ zeros on $[z_0, b]$. If $i = 0$ and $z_0 < x_0$, then (2.3), (2.4), and (2.5) show that h has at least $k + 1$ zeros on $[z_0, b]$.

If $i = 1$ and $a = x_0 = z_0$, then $h(a) = (-1)^k + (-1)^{k-1} = 0$; if $x_0 < z_0$ then (2.3), (2.4), and (2.5) show that h has at least one zero on $[x_0, z_0]$. Thus (2.7) is true if $i = 1$. If $i > 1$, (2.7) follows easily from (2.3), (2.4), and (2.5); and (2.8) follows just as in the polynomial case [2]. This completes the proof of (2.1). In addition, (2.2) also follows from the proof in [2] and the proof of the theorem is complete.

References

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